Miss Lee, a special education teacher at Hillside Elementary School, just finished her first mathematics lesson on regrouping. While reviewing her students’ work, she noticed some students failed to regroup accurately, other students were unable to determine when regrouping was needed and regrouped in every problem, and a few students were unable to finish all the problems in the given lesson time. As Miss Lee continued to examine her students’ work, she realized that some of her students needed assistance with conceptual knowledge, others needed assistance with procedural knowledge, and others needed assistance with declarative knowledge. With this in mind, Miss Lee began to plan several lessons related to each of these knowledge types.

Many students with learning difficulties in mathematics struggle to master computation skills (Bryant, Bryant, Gersten, Scammacca, & Chavez, 2008; Gersten, Jordan, & Flojo, 2005). Students who fail to develop proficient skills and knowledge related to the four computational operations (i.e., addition, subtraction, multiplication, division) in the early grades are likely to experience mathematics difficulties as they attempt to progress through the mathematics curriculum (Calhoon, Wall, Flores, & Houchins, 2007). Thus, it is very important to provide high-quality computation instruction to ensure early mastery of these foundational skills.

As teachers plan their computation curriculum, it is important to balance instruction designed to address three types of mathematical knowledge: conceptual, procedural, and declarative (Miller & Hudson, 2007). Conceptual knowledge involves a deep understanding of the meaning of mathematical operations as well as an understanding of the relationships and connections among the operations. Conceptual knowledge results in the ability to generalize and apply mathematics understanding to various situations and settings. Procedural knowledge involves the ability to solve mathematics problems using a step-by-step process that ultimately results in an accurate solution. Declarative knowledge involves the ability to memorize information that is factual in nature (e.g., looking at a number and knowing its name, looking at a math fact and immediately knowing the answer without having to figure it out). Proficiency in all three knowledge types enhances students’ computation competence and increases the likelihood of success in other areas of mathematics.

Typically, practices designed to develop conceptual understanding are provided during initial instruction on a specific skill (e.g., addition without regrouping). Next, instruction shifts to the development of procedural understanding, and finally, emphasis is placed on the development of declarative knowledge of the same skill. As instruction proceeds to a higher level skill (e.g., addition with regrouping), conceptual understanding is again emphasized first. Instruction then progresses to the development of procedural and finally declarative knowledge related to the new skill (Hudson & Miller, 2006). Although teachers frequently plan their mathematics
instruction using this sequence (i.e., conceptual, procedural, declarative knowledge), it is important to note that the development of conceptual knowledge enhances the development of procedural knowledge and vice versa (National Mathematics Advisory Panel, 2008; Rittle-Johnson, Siegler, & Alibali, 2001). Similarly, declarative knowledge with basic addition and subtraction facts supports procedural knowledge related to higher level skills (e.g., addition and subtraction with regrouping and problem solving; Botzge, 2001).

Because the development of declarative knowledge related to computational skills takes time and a significant amount of practice, it makes sense for teachers to begin conceptual level instruction on a new computational skill while their students continue to work on declarative knowledge of a skill not yet mastered. For example, students who have conceptual and procedural understanding related to addition without regrouping may need to continue fluency practice to develop declarative knowledge related to this skill while conceptually-based instruction related to addition with regrouping begins.

This article provides examples of evidence-based practices designed to help students develop conceptual, procedural, and declarative knowledge specifically related to mathematics computation. Although specific examples are provided related to each practice, it is important to note that the practices are not necessarily limited to the skills used in the examples.

**Developing Conceptual Knowledge in Computation**

The concrete-representational-abstract (CRA) teaching sequence helps students gain conceptual understanding related to addition, subtraction, multiplication, and division. CRA instruction begins with the use of manipulative devices (i.e., concrete), progresses to the use of pictures or tallies (i.e., representational), and ultimately involves solving problems using numbers only (i.e., abstract). Educators and researchers have documented the value of developing mathematical models using both manipulative devices and pictorial representations (Flores, 2009; Gersten et al., 2009; Ketterlin-Geller, Chard, & Fien, 2008; Miller & Kaffar, 2011) in conjunction with explicit instructional procedures. Explicit instruction is clear, accurate, and unambiguous (Stenz, Kinder, Silbert, & Carnine, 2006) and frequently includes teacher demonstrations with verbal descriptions, guided practice (i.e., high levels of support initially with systematic decreases in support as student proficiency increases), and independent practice.

When providing explicit instruction that involves the use of manipulative devices, it is important to note that teacher demonstrations also include presentation of the numerical problem. Thus, students see the abstract problem (i.e., number representation) and a concrete model of the problem (i.e., 3-dimensional representation) simultaneously. This facilitates students' understanding of what the abstract problem actually means (e.g., addition means two quantities put together and results in a larger amount). See Figure 1 for teacher demonstration examples that involve the use of explicit instruction and manipulative devices to promote conceptual understanding in computational skills.

When providing explicit instruction that involves the use of manipulative devices, teacher demonstrations should also include presentation of the numerical problem.

After teacher demonstrations, students are given opportunities to represent and solve similar problem types using manipulative devices through guided practice that involves teacher prompts and verbal cues. As students demonstrate success with guided practice, they are provided opportunities for independent practice (i.e., solving problems without teacher assistance). Following several lessons using manipulative devices, pictorial models are presented. First, the teacher demonstrates how to solve several problems using pictures and/or tallies and then students are given opportunities with numerical representations of the problem will help Miss Lee's students understand why the number in the tens column is changed (i.e., trading a ten for 10 ones) and why the addition of 10 ones now makes subtraction possible (see Figure 2). After teaching students how to demonstrate regrouping using base 10 blocks, Miss Lee progresses to representational-level instruction using pictures to demonstrate the regrouping process (see Figure 3). While Miss Lee provides regrouping instruction at the concrete and representational levels, she includes some problems that do not...
**Addition Instruction**

**Step One**

\[
\begin{array}{c}
3 \\
+2
\end{array}
\]

**Step Two**

\[
\begin{array}{c}
3 \\
+2
\end{array}
\]

**Step Three**

\[
\begin{array}{c}
3 \\
+2
\end{array}
\]

Teacher Script: To solve addition problems using cubes, I look at the first number, 3, and count that many cubes. One, two, three. Then, I look at the second number, 2, and count that many cubes. One, two. Now, I count all the cubes together to get my answer. One, two, three, four, five. I write five in the answer space.

**Subtraction Instruction**

**Step One**

\[
\begin{array}{c}
3 \\
-1
\end{array}
\]

**Step Two**

\[
\begin{array}{c}
3 \\
-1
\end{array}
\]

**Step Three**

\[
\begin{array}{c}
3 \\
-1
\end{array}
\]

Teacher Script: To solve subtraction problems using cubes, I look at the first number, 3, and count that many cubes. One, two, three. Then, I look at the second number, 1, and take that many away. Now, I count the cubes that are left. One, two. I write two in the answer space.

**Multiplication Instruction**

**Step One**

\[
2 \times 3
\]

**Step Two**

\[
2 \times 3
\]

**Step Three**

\[
2 \times 3
\]

Teacher Script: To solve multiplication problems using cubes, I look at the first number, 2, and count that many paper plates to represent the groups. One, two. Then, I look at the second number, 3, and put that many objects in each group. One, two, three. One, two, three. Now, I count all the cubes together. One, two, three, four, five, six. I write six in the answer space. Two groups of three equal six.

**Division Instruction**

**Step One**

\[
8 \div 2 =
\]

**Step Two**

\[
8 \div 2 =
\]

**Step Three**

\[
8 \div 2 =
\]

Teacher Script: To solve division problems using cubes, I look at the first number, 8, and count that many cubes to represent the total number of cubes. One, two, three, four, five, six, seven, eight. Then, I look at the second number, 2, and count that many paper plates to represent the groups. One, two. Next, I put one cube in the first group and one cube in the second group, another cube in the first group and another cube in the second group. I continue until all cubes are placed in the groups. Now, I count to see how many cubes are in each group. One, two, three, four. I write four in the answer space. Eight cubes divided into two groups equals four cubes in each group.
Developing Procedural Knowledge in Computation

A variety of cognitive strategies have been developed to assist students with the development of procedural knowledge related to addition, subtraction, multiplication, and division. Cognitive strategies are designed to help students remember the step-by-step process involved in solving computation or word problems. Regardless of whether the strategy is designed for computation or word problems, two characteristics should be present. First, the strategy steps should be applicable to many problems to ensure that the time spent learning the strategy results in greater efficiency in solving more than just a few problems. Second, each strategy step should prompt the student to perform an overt action, such as write the answer in the answer space; use a cognitive or metacognitive technique, such as paraphrase the problem question; or apply a rule, such as use the rounding rule (Hudson & Miller, 2006). Cognitive strategies used to solve computation problems sometimes involve a first-letter mnemonic device (e.g., each letter in a word represents the first letter in the steps used to solve the problem). See Tables 1 and 2 for mnemonic devices that have been used successfully to help students solve basic facts and regrouping problems.

Other memory tools include the use of key letters, key words, or key sentences to help students remember procedural steps for solving problems. For example, the 4 Bs Strategy (Brown &
Frank, 1990) helps students solve subtraction problems with regrouping. The four Bs are Begin, Bigger, Borrow, and Basic Facts. Key word and key sentence strategies have been used to help students remember the procedural steps for solving long division problems. For example, the key words Daddy, Mother, Sister, and Brother serve as cues for Divide, Multiply, Subtract, and Bring Down (Mercer & Mercer, 2010). Similarly, the key sentence, “Does McDonald’s Sell Cheese-Burgers?” helps students remember the steps in long division (i.e., Divide, Multiply, Subtract, Compare, Bring down; Bley & Thornton, 2001).

When teaching students to use cognitive strategies or memory tools, it is important to provide multiple opportunities for students to memorize the mnemonic device steps, key letters, key words, and/or the sentences. Students must learn the memory tool with automaticity in order to apply it successfully to the mathematics problems they are trying to solve.

The addition with regrouping mnemonic device (i.e., RENAME) presented in Table 2 can be adapted to help Miss Lee’s students who fail to regroup accurately when solving subtraction problems. The adapted RENAME steps for subtraction are: (a) Read the problem, (b) Examine the ones column using the BBB phrase (i.e., Bigger number on Bottom Break down ten and trade), (c) Note ones in the ones column, (d) Address the tens column, (e) Mark tens in the tens column, and (f) Examine and note hundreds; exit with a quick check (i.e., add product to the subtrahend to see if the total is equal to the minuend). This mnemonic device is especially helpful for students who have difficulty remembering the correct sequence of the steps involved in solving subtraction problems that require regrouping and/or for students who forget to perform certain steps.

**Figure 3. Representational-Level Instruction Related to Subtraction With Regrouping**

**Representational Instruction**

1. Look at the first number and check to be sure the drawing represents the first number.

2. Look at the ones column. We can't take 7 ones from 3 ones. So we trade 1 ten for 10 ones.

3. Take 7 ones away from 13 ones, count how many ones are left, and write in ones column.

4. Look at tens column and take 1 ten from 2 tens, count how many tens are left, and write in tens column.

---

**Developing Declarative Knowledge in Computation**

Implementation of 1-minute computational probes is one of the most effective instructional strategies for helping...
Table 1. Mnemonic Device for Addition, Subtraction, Multiplication, and Division Facts

<table>
<thead>
<tr>
<th>Strategy Steps</th>
<th>Student Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discover the sign.</td>
<td>Student looks at the sign to determine whether the problem requires addition, subtraction, multiplication, or division.</td>
</tr>
<tr>
<td>Read the problem.</td>
<td>Student reads the problem aloud or silently.</td>
</tr>
<tr>
<td>Answer, or draw and check.</td>
<td>Student tries to think of the answer, but if the answer is not yet memorized, the student draws tallies to figure out the answer. Student checks to be sure the correct number of tallies were drawn and counted. (This step assumes that students have been taught how to represent and solve problems using tallies.)</td>
</tr>
<tr>
<td>Write the answer.</td>
<td>Student writes the answer in the answer space.</td>
</tr>
</tbody>
</table>


students develop declarative knowledge. Typically, probe sheets consist of problems from the same skill set (e.g., subtraction) and have more problems than a student can answer in the allotted amount of time of 1 minute. All students have a copy of their probe sheet face down on their desks. The teacher says, “Get ready. Please begin.” At this cue, students turn their probe sheet over and begin working the problems moving from left to right. They answer as many problems as they can completing the same subtraction probe sheet, the teacher and students can read the problems and answers together in sequential order with a slight pause in between problems to allow students to mark correct and/or error responses (e.g., 17 minus 9 equals 8—pause—15 minus 6 equals 9). If students are completing different probes (e.g., some are completing addition probes while others are completing subtraction probes), the teacher may have students use calculators or answer keys to check their work. Another option is for teachers to do the checking immediately following the

Implementation of 1-minute computational probes is one of the most effective instructional strategies for helping students develop declarative knowledge.

Table 2. Mnemonic Device for Addition With Regrouping

<table>
<thead>
<tr>
<th>Strategy Steps</th>
<th>Student Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Read the problem.</td>
<td>Student reads the problem aloud or silently.</td>
</tr>
<tr>
<td>Examine the ones column: 10 or more go next door.</td>
<td>Student adds the numbers in the ones column. If the sum is 10 or more, the student records the “1” crutch number in the tens column.</td>
</tr>
<tr>
<td>Note ones in the ones column.</td>
<td>Student records the ones in the answer space of the ones column.</td>
</tr>
<tr>
<td>Address the tens column: 10 or more go next door.</td>
<td>Student adds the numbers in the tens column. If the sum is 10 or more, the student records the “1” crutch number in the hundreds column.</td>
</tr>
<tr>
<td>Mark tens in tens column.</td>
<td>Student records the tens in the answer space of the tens column.</td>
</tr>
<tr>
<td>Examine and note hundreds; exit with a quick check.</td>
<td>Student adds and writes the number of hundreds in the hundreds column. Student then re-adds the problem to be sure the answer is correct.</td>
</tr>
</tbody>
</table>

What Does the Research Say?

Implement Explicit Mathematics Instruction

Baker, Gersten, and Lee (2002) conducted a meta-analysis that involved 15 high-quality studies related to mathematics instruction. Their findings revealed moderate to strong effects for the use of explicit instruction for teaching mathematics to high-risk students. Similarly, Krosbergen and Van Luit (2003) completed a meta-analysis that involved 58 studies related to mathematics interventions for elementary students. Their findings revealed that explicit instruction was more effective than less structured reform-based methods for teaching basic mathematics skills to students with mathematics difficulties.

Use Concrete and Representational Models

Research related to the use of concrete and representational models to promote conceptual understanding reveals acquisition benefits for students with learning disabilities, students with intellectual disabilities, and students who are low performers in mathematics computation (Gersten et al., 2009). The concrete–representational–abstract teaching sequence combined with explicit instructional procedures has been used to present systematic and effective conceptual models for addition, subtraction, multiplication, and division (Carmack, 2011; Ferreira, 2009; Flores, 2009; Miller & Kaffar, in press, 2011; Miller & Mercer, 1998; Morin & Miller, 1998).

Teach Cognitive Strategies

Results related to teaching students with learning difficulties how to use a variety of mathematics cognitive strategies have been very positive and meet rigorous criteria used to identify evidence-based practices (Montague, 2008). In the previously noted CRA literature, cognitive strategies were used to facilitate the transition between representational and abstract computation instruction and facilitated independent problem solving among the students. Additional research related to cognitive strategies focuses on computation embedded within word problems (e.g., Chung & Tam, 2005; Montague, 2007, 2008) and indicates benefits for students with mild disabilities, students with learning disabilities, and students with mild intellectual disabilities.

Administer Mathematics Timings

Research related to the effects of using mathematics timings to help students become more fluent in their computational abilities reveals positive outcomes. Specifically, the use of timings has been successful for developing declarative knowledge in whole number operations (Le Grice, Mabin, & Graham, 1999), multiplication (Chapman, Ewing, & Mozzoni, 2005), and division (Chiesa & Robertson, 2000).

Both timed probes and instructional games that include a timing element can be used to help Miss Lee's students (i.e., case presented at the beginning of this article) who fail to complete all of the independent practice problems within the allotted lesson time. If the reason the students fail to finish the subtraction with regrouping problems is because they do not have basic facts (i.e., single-digit problems) memorized, then the timed probes and games should involve basic facts. If, however, the student is fluent with the basic facts, but is very slow remembering the regrouping process, then the timed probes and games should involve 2-digit problems that require regrouping.

Final Thoughts

Students who struggle with basic computation skills need balanced mathematics curricula and instruction that addresses conceptual, procedural, and declarative knowledge. This type of instruction helps ensure that students (a) understand what the specific operations mean, (b) acquire the ability to follow step-by-step processes for obtaining accurate answers, and (c) become fluent when solving problems. The evidence-based practices described in this article (i.e., CRA, cognitive strategies, 1-minute timings) assist teachers in providing high-quality, explicit computation instruction (See box, “What Does the Research Say?”). Providing this type of instruction helps establish a solid foundation on which additional mathematics learning can
occur. The hierarchical nature of mathematics necessitates this solid foundation for students to experience success in mathematics throughout their school careers and beyond.

References


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